The Nature of Mathematics:
Let's Talk about It

Teachers can offer opportunities for K–12 students to reflect on the nature of mathematics (NOM) as they learn.

Lucy A. Watson, Christopher T. Bonnesen, and Jeremy F. Strayer

Most mathematics teachers have an answer for the following student question. Some teachers like answering it; others hate it: “Why do I need to know ______?”. Fill in the blank with any mathematical concept or procedure, and we can all imagine students asking it. So, what is your answer? Some answers that teachers give focus on practical applications to science, technology, engineering, and mathematics education fields or different real-life scenarios. Other answers bring out the beauty and wonder in mathematics. Whatever your response, we, the authors, contend that every answer to this question is influenced by the answerer’s view of the nature of mathematics (NOM).

What should teachers help students come to understand about NOM? The standards, documents, and recommendations that guide the mathematics education profession are motivated by a relatively common view of NOM, but these documents do not clearly communicate what that view is. Even more, they do not give recommendations for what we should be teaching our students concerning NOM. In this article, we present a brief description of the different views of NOM, share a five-point view of NOM that undergirds our profession’s
Appendix 2 (continued)

guiding documents, and describe ways of providing opportunities for teachers and students to have conversations in the classroom that build understanding of NOM.

**VIEWS OF THE NATURE OF MATHEMATICS**

Historically, that different people have different views about mathematics is widely accepted. That is to say, if you posed the question, “What is mathematics?” to students, teachers, or mathematicians, you will likely receive different answers (Dossey 1992). From a teaching perspective, considering different views is critical, because a teacher’s view of NOM can influence instruction in the classroom and potentially students’ views of NOM. Thompson (1992) provided three descriptions of NOM as it relates to teachers’ views. First, teachers can view mathematics as a dynamic, problem-driven discipline in which NOM is defined as being creative, open to revision, and a product of creativity and inquiry. Second, teachers may hold the view that mathematics is a static, unified body of knowledge in which NOM is defined as being bound by truths that are discovered and never change. Finally, teachers may see mathematics as a bag of tools in which NOM is defined as a set of rules and facts to be memorized. Imagine three different classrooms, each with a different teacher, and each teacher with a different one of these three views of NOM. In these three classrooms, the approach to teaching mathematics will be different because these teachers hold different views of NOM. Also, what students are learning about NOM will be different because the students are exposed to very different approaches to mathematics. For example, a teacher who holds the view that mathematics is a bag of tools will likely use lecture, example, and practice as a main teaching strategy, producing students who hold the belief that mathematics is computation. Whereas a teacher who holds the view that mathematics is a dynamic, problem-solving discipline will allow students to explore the mathematics, provide an argument, and then engage in discussion allowing students opportunities to build their understanding without being told exactly how to proceed. These different, often opposing, views of NOM produce different classrooms and different learning outcomes.

**NATURE OF MATHEMATICS AND THE STANDARDS**

Although individuals can hold different views regarding NOM, the mathematics education community can look to the professional standards for insight into how to attend to the nature of the subject we teach. While providing insight into the effective teaching and learning of the subject, mathematics education standards contain no explicit guidance for teaching and learning NOM. Perhaps it is because there was broad agreement on NOM from the start, or maybe it was an effort not to perpetuate the Math Wars, but mathematics educators decided not to give guidance on teaching NOM in the guiding documents that have shaped our profession for the last 30 years. We think this should change, and we detail what we see as a relatively common view of NOM in these documents.

**The Five-Point View of the Nature of Mathematics**

We can see how a particular view of NOM has influenced the field of mathematics education by exploring our field’s foundational documents: NCTM’s (2000) Process Standards, the Common Core’s (NGA Center and CCSSO 2010) Standards for Mathematical Practice (SMP), NCTM’s (2014) Mathematics Teaching Practices...
(MTPs), the Association of Mathematics Teacher Educators (AMTE 2017) standards for preparing teachers, and the National Research Council’s (NRC 2001) strands for mathematical proficiency. These guiding documents are based on research, describe functional ideas about mathematics, and are central in their influence of teachers and students. However, they focus on ideas about the teaching, learning, and the discipline of mathematics together. For example, the Process Standards and SMF focus on actions of students doing mathematics in a classroom, MTPs focus on strategies for teachers, AMTE’s standards focus on programs, and the strands for mathematical proficiency relay outcomes for students who have learned mathematics successfully.

A further analysis of these documents by Watson (2019) resulted in a list of commonalities regarding NOM (see figure 1). We propose using this list as a foundation for providing opportunities for students to learn about NOM.

To understand the purpose of the five-point view of NOM, distinguishing it from the standards and professional documents from which they are derived is important. The standards documents offer details about how to effectively teach mathematics (MTPs; NCTM 2014), what students should be doing when learning mathematics (SMF; NGA Center and CCSSO 2010), characteristics that create a productive mathematics classroom (Process Standards, NCTM 2000), and characteristics of mathematically proficient students (Strands for Mathematical Proficiency, NRC 2001). The five-point view combines what is implied within these standards documents to explicitly state the characteristics of mathematics as a discipline (i.e., NOM) and not necessarily how it is related to the teaching and learning of mathematics. See video 1 for more details about this distinction.

Although the five-point view is distinct from standards documents, noting the connections among them can be useful when considering how the five-point view can be used in classroom instruction. For example, let’s specifically consider connections with SMF. When students look for and express regularity in repeated

![Video 1 How Are the Nature of Mathematics and Professional Standards Different?](image)

Watch the full video online.

<table>
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<th>Fig. 1</th>
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<td><strong>The Five-Point View of NOM</strong></td>
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<tr>
<td>1. Mathematics is a product of the exploration of structure and patterns.</td>
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<tr>
<td>2. Mathematics uses multiple strategies and multiple representations to make claims.</td>
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<tr>
<td>3. Mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others.</td>
</tr>
<tr>
<td>4. Mathematics is refined over time as cultures interact and change.</td>
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<tr>
<td>5. Mathematics is worthwhile, beautiful, often useful, and can be produced by each and every person.</td>
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The five-point view of NOM is distilled from the field’s foundational documents.
reasoning, they have the opportunity to understand that mathematics is a product of the exploration of structure and patterns and is refined over time as cultures interact and change. When students make sense of problems and persevere in solving them, they have the opportunity to develop the idea that mathematics is worthwhile, beautiful, and often useful and can be produced by each and every person. When students model with mathematics, they have the opportunity to understand that mathematics uses multiple strategies and representations to make claims and that mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others. In fact, modeling tasks usually have opportunities to promote all five of the characteristics of NOM.

We believe that teachers can use the five-point view of NOM to help each and every student refine and deepen their understanding of NOM. In the remainder of this article, we share a teaching strategy for using this list while teaching with existing rigorous mathematical activities to help students build well-rounded understandings of NOM. As we describe these lessons, we offer a shared language for discussing NOM with the hope that teachers will use these strategies to draw students’ attention to NOM and create opportunities for them to refine and deepen their understandings of NOM.

**TEACHING AND LEARNING THE NATURE OF MATHEMATICS**

We begin this section by describing a four-part strategy (see figure 2) for how teachers and students can reflect on NOM while completing mathematics tasks. Then we show how that strategy can be enacted during mathematics lessons at four different grade levels. The focus in all instances is on the strategy for explicitly reflecting on NOM.

When using this teaching strategy, the teacher must first choose a rich mathematics task with the potential to bring out aspects of the five-point view categories and that aligns with their mathematical goals. Prior to the lesson or unit, the teacher should make notes regarding characteristics of the five-point view that the task allows the teacher to emphasize. Second, at the beginning of a lesson or unit, the teacher provides an opportunity for students to explicitly reflect on their own views of NOM through a prompt (such as a think-pair-share or an entrance ticket). Third, as students complete different mathematical tasks, the teacher may wish to prompt students periodically to reflect on their views of NOM. Only prompt NOM reflection if it is natural—do not force it—as students can tire of NOM conversation if it is a constant focus. Fourth and finally, the teacher should share or reshare the five-point view of NOM and ask students to reflect on how, if at all, their views of NOM have deepened or changed since the beginning of the lesson.

**Fig. 2**

<table>
<thead>
<tr>
<th>Four-Part Strategy for Teaching NOM</th>
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<tbody>
<tr>
<td>1. Select mathematical tasks that will promote one or more aspects of the Five-Point View of NOM.</td>
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<tr>
<td>2. Provide students with an opportunity to reflect on and communicate their personal views of NOM at the beginning of a lesson or unit. Consider sharing the Five-Point View here if students need a starting point.</td>
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<td>3. Launch the lesson or unit, monitor students’ progress as you normally would, and note statements and student work that relates to NOM. If a natural moment arises, ask students to reflect on their NOM views as they complete tasks.</td>
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<tr>
<td>4. At the close of a lesson or unit, share the five-point view and provide students with an opportunity to reflect on and refine their views of NOM.</td>
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This four-part strategy for teaching NOM is useful for providing opportunities for students to reflect on NOM.
or unit. As in the beginning of the strategy, students should share their thinking through a discussion, exit ticket, or some other means.

Research in science education has taught us that students must have opportunities to explicitly reflect on aspects of the nature of science while engaging in science practices to develop understandings of the nature of science (Schwartz, Lederman, and Crawford 2004). Incorporating the language of NOM into our everyday, common, good teaching practices is important because we are bringing a shared language to students and teachers alike in what otherwise would remain unnamed. Although this four-part strategy seems similar to everyday, good, common teaching practices, the intentional addition of NOM and the importance of explicit reflection set it apart. Furthermore, the four-part strategy emphasizes the time needed for students to deepen their understanding of NOM—it can start with a lesson, but it should be spoken of and refined over multiple lessons, units, semesters, or years.

We now briefly share examples of how this four-part strategy may be enacted during lessons at several grade levels. Any high quality task can be used for these lessons, and we chose four from NCTM’s (2020) collection of activities with rigor and coherence (ARC: counting strategies (K), equivalent fractions (grade 3), discovering area relationships (grade 6), and absolute value (high school algebra). For more mathematical details about each activity, follow the provided links. We present short descriptions of the lessons, drawing on our collective experiences with teachers and students to provide insights into how those enactments may unfold in varying grade levels. Although four examples are provided, the reader should focus on the example(s) that are of interest based on grade level. It is not necessary to read all four.

- For an example of this NOM teaching strategy in a PK–2 lesson, go to the section below.
- For an example of this NOM teaching strategy in a 3–5 lesson, go to p. 357.
- For an example of this NOM teaching strategy in a middle school lesson, go to p. 357.
- For an example of this NOM teaching strategy in a high school lesson, go to p. 358.

Counting in Kindergarten ARC Activity

The mathematical goal of this lesson is to have students make connections among their understandings of quantity and numeral. As teachers plan, they may notice how even the idea of a number is like a pattern.

If one has four snap cubes, four crayons, four shells, and so on, then in all cases, the quantity itself is like a pattern that points to mathematical structure (statement 1 of the five-point view), regardless of substance or shape. Likewise, the fact that each number can be represented by a quantity of several objects, or even as a point on a number line points to multiple representations (statement 2 of the five-point view). Moreover, students and the teacher can work collaboratively in this lesson to critique and verify claims, such as explicitly counting out loud to prove which symbol goes with which quantity, in alignment with statement 3. Finally, this activity is accessible to all students because of its concrete, hands-on nature, in alignment with statement 5. Therefore, this lesson is ideal for explicitly reflecting on NOM with students.

The second part of the strategy is to offer students a prompt to reflect on NOM just prior to completing the activity in class. For kindergartners, we want to keep in mind that students will respond in concrete ways. When the teacher begins by asking, “What is math?” students may respond by saying numbers, counting, telling time, matching patterns, or by drawing on other subjects they are learning. We can see these ideas expressed in an interview with a six-year-old student when we asked her, “What is math?” (see video 2). She provides a concrete idea of what she thinks mathematics is by stating, “Math is like this, two minus five.
Appendix 2

equals what?” She also describes mathematics as patterns when she tells us about the upper- and lower-case letters. Though this student is at the beginning of her mathematics journey, she is able to describe what she thinks mathematics is, and more importantly, she is being asked that question. If teachers continue to ask her the same question, what is mathematics, over the years, she will have abundant opportunities to reflect on and refine her view of NOM.

Next, the teacher implements the activity, giving students an opportunity to engage in mathematical thinking so they can achieve the goals of the lesson. Finally, the teacher will debrief with students at the end of the lesson by helping them see how they engaged in aspects of the five-point view using concrete terms that their kindergartners can understand. They may help students verbalize that their experiences showed that mathematics is about numbers, counting, finding how many, seeing patterns, and other ideas. In this way, teachers and students can form the habit of thinking “What is that I’m doing?” at a very young age.

- For a grades 3-5 example, go to the section below.
- For a middle school example, go to the middle of the next column.
- For a high school example, go to p. 338.
- To go to the discussion, go to p. 339.

Equivalent Fractions in Grade 3 ABC Activity
This four-part lesson focuses on the concept of fraction, comparing fractions, and determining equivalent fractions. When planning to use this lesson for reflecting on NOM, the teacher may first observe several connections to the concept of structure. Students learn about the overall structure of a fraction and explore the meanings of and relationships among the numerator, denominator, and whole unit. Students also use Cuisenaire® Rods to better understand how the size of a fraction is always relative to the whole unit, and they learn how to connect the Cuisenaire Rods to the symbols for fractions and the number line. Taken together, the experiences in this lesson offer opportunities for students to develop understanding of the structure of fractions (statement 1 of the five-point view), use multiple strategies and multiple representations (statement 2), and collaborate in a hands-on setting to critique and justify their claims (statement 3) while producing their own mathematical knowledge (statement 5). Therefore, this activity is a great opportunity for students to reflect on NOM.

Just prior to beginning this activity, students will need to consider a prompt such as “In your own words, what is mathematics?” This would be a great prompt for a written mathematics journal so that students could revisit their responses and keep a record of how their thinking has changed from time to time throughout the year. Next, students complete the activity in class. The teacher may wish to pick and choose parts of this thorough, four-part activity, on the basis of students’ prior knowledge and the teacher’s instructional goals. At the conclusion of the activity, the teacher displays and reads the list of NOM statements, asks students to consider their personal NOM statements from prior to the lesson, and asks students to write down how their views of what mathematics is have changed. Ideally, this process will engender sophisticated views of NOM beyond the idea that mathematics is about numbers, formulas, following steps, getting answers, and the like.

- For a PK-2 example, go to p. 356.
- For a middle school example, go to the section below.
- For a high school example, go to p. 358.
- To go to the discussion, go to p. 359.

Discovering Area Relationships in Middle School

ABC Activity
This lesson uses several examples of rectangles, triangles, and parallelograms to help students see the underlying structure and patterns that give rise to the area formulas for those shapes, in accordance with statement 1 of the five-point view. Additionally, multiple representations are emphasized in the lesson, which prompts students to find areas using many different strategies, in support of statement 2 of the five-point view. Finally, the lesson provides opportunities for students to communicate their ideas to others as they work in groups and contribute during whole-class discussions (statement 3 of the five-point view). In these ways, students have the potential to produce mathematical knowledge due to the hands-on and concrete nature of the activity, especially during the initial phases (statement 5 of the five-point view).

In part 2 of the teaching strategy, the teacher should offer students an opportunity to reflect on their view of what mathematics is. The teacher may ask for a response to the following question through an entrance ticket: Different people describe mathematics in different ways. In your own words, answer the question “What is
Students may draw on their own experiences when initially asked, so potential answers may include the following: numbers, shapes, solving problems, or equations.

In part 3 of the strategy, the teacher must encourage students to explore the conceptually connected tasks in small groups and participate in brief whole-class discussions after each task. By using multiple strategies, patterns, and representations, students will have opportunities to build a deep understanding of how the area formulas for triangles and rectangles relate. This will prepare them to solve area problems with all types of triangles, including those with heights that are difficult to detect. Finally, students can consider all of these strategies together to explore the area formula for parallelograms.

At the conclusion of the lesson, the teacher may choose to share the five-point view statements with the class and give students an opportunity to briefly discuss their reactions. One possible ending is to use another prompt through an exit ticket: Think again about how you would describe what mathematics is in your own words. What would you change, add to, or further clarify in what you wrote at the beginning of class? Expand on your thoughts.

We enacted an adaptation of the grade 6 lesson in a geometry course for preservice elementary school teachers. You may view this lesson and student responses in video 3.

- For a PK-2 example, go to p. 356.
- For a grades 3–5 example, go to p. 357.
- For a high school example, go to the top of the next column.
- To go to the discussion, go to p. 359.

**Video 3 An Adapted Grade 6 Lesson for a Geometry Class**

[Watch the full video online.]

**Absolute Value in High School ARC Activity**

When going through part 1 of the four-part strategy, the teacher will see that this lesson on absolute value and functions highlights the categories of patterns, structure, multiple strategies, justification, beauty, and multiple representations, in alignment with statements 1, 2, 3, and 5 of the five-point view of NOM. The lesson begins by asking students to use a Dyanograph applet (which shows different linear and absolute value function relationships by dynamically linking inputs to outputs on two separate number lines) to determine, express, and justify their mathematical description of those functions. Students continue to discover patterns and develop a deeper understanding of the structure of the idea of absolute value by comparing several graphs. The lesson ends by asking students to build understanding of what it means to solve absolute value equations for $x$ by using Dyanographs, traditional function notation, and traditional graphs. The lesson gives students opportunities to use multiple representations that beautifully connect function representations to build understandings of the similarities and differences between several forms of linear and absolute value functions. Therefore, this activity is excellent for exploring NOM through the focus of the five-point view.

In part 2 of the NOM strategy, the teacher should offer an opportunity for students to reflect on their view of what mathematics is. Because this lesson involves functions, graphs, and equations, it may be advisable to ask the more specific entrance prompt: "In your own words, describe what algebra is." Students often draw on their recent experiences when initially answering NOM prompts, so potential answers may include mathematics with letters in it, equations, or solving for $x$.

The teacher should implement the lesson, encouraging students to explore and share their mathematical thinking in the third part of the NOM teaching strategy. In part 4, the teacher should ask students to reflect on the five-point view and then communicate their thoughts during a final prompt such as Think again about how you would describe what algebra is in your own words. What would you change, add to, or further clarify in what you wrote at the beginning of class? Expand on your thoughts. The teacher may see that students expand their notion of algebra to include relationships between inputs and outputs, connections between formulas and graphs, or multiple representations of functions. In this way, NOM prompts can be more specific, focusing on the nature of a particular subject within mathematics.
Appendix 2 (continued)

DISCUSSION
We believe that each of the preceding lessons representing existing NCTM materials can promote aspects of NOM, though they do not currently do so explicitly. Our goal is to reinvent the wheel but to show teachers and the education community at large that rigorous, meaningful tasks can allow students to easily attend to aspects of NOM. These tasks not only allow students to work together, explore the mathematics, and share their thinking but also offer good questions teachers can ask students that will help promote ideas about NOM. To illustrate this idea further, in figure 3, we present a few of the questions that are included in the lesson plans from the ARCs we used in the earlier tasks (K, grade 3, grade 6, and high school) and connect them to how these questions can open space in a classroom for reflection on NOM. Each of these questions serves as an example of the type of question that can help advance one’s views of NOM because the questions draw attention to specific characteristics from the five-point view of NOM.

Asking the types of questions in the center column of figure 3 alone is not enough to guarantee that students advance their view of NOM, just as students engaging in the SMP is insufficient on its own. The combination of these already high-quality tasks coupled with the practice of sharing the language of NOM and asking students to explicitly reflect on NOM can bring about refinement of students’ ideas regarding the NOM.

CONCLUSION
In this article, we demonstrate how to teach lessons that give students opportunities to refine and deepen their understandings of NOM while they learn mathematics at multiple grade levels. All that teachers need is the five-point view of NOM, a high-quality mathematics task to implement (e.g., ARGs), and prompts for explicit reflection on NOM (e.g., entrance and exit slips). For NOM reflections to meaningfully affect students’ understandings of mathematics will require repetition over time—preferably over multiple years. As the student reflections from the grade 6 lesson showed earlier, progress is incremental and unfolds slowly.

We note here that as teachers and students engage in these NOM reflective practices, they may discover the need to add to the five-point view because NOM means additional things within their learning community. Classes’ understandings of NOM should reflect what mathematics means according to their experiences and contexts. Teachers can use additions judiciously to further help students understand that mathematics is a growing, vibrant, and creative space that is open to each and every person.

Finally, as students refine and deepen their understandings of NOM, perhaps they will ask the question “Why do I need to know _____?” less frequently. Or, perhaps they will develop their own insightful answers to this question on the basis of a deepened understanding that exploration, structure, strategy, justification, communication, refinement, and beauty are at the heart of mathematics. Understanding that mathematics is about more than application to other fields will serve students well. Regardless, we hope that this article helps everyone in the mathematics education community take up a shared language with which to discuss and build understanding together of NOM.
## Appendix 2 (continued)

**Fig. 3**

<table>
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<tr>
<th>Lesson</th>
<th>Question Asked in ARC</th>
<th>Connection to the Five-Point View</th>
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<tbody>
<tr>
<td><strong>Kindergarten</strong></td>
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<tr>
<td></td>
<td>A. If we rearrange the cubes and count again, what will happen to the total?</td>
<td>A. Mathematics is a product of the exploration of structure and patterns.</td>
</tr>
<tr>
<td></td>
<td>B. What number is one more (one less) than this? How do you know?</td>
<td>B. Mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others.</td>
</tr>
<tr>
<td></td>
<td>C. What is the connection between the last number name said and the number of objects counted?</td>
<td>C. Mathematics is a product of the exploration of structure and patterns.</td>
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<tr>
<td><strong>Grade 3</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A. What fraction name can we give each piece? Why do we give it that name?</td>
<td>A. Mathematics is refined over time as cultures interact and change.</td>
</tr>
<tr>
<td></td>
<td>B. How do the unit fractions we created using the rods help us understand numbers less than one whole? Will these numbers help us solve problems in our everyday lives?</td>
<td>B. Mathematics is worthwhile, beautiful, often useful, and can be produced by each and every person.</td>
</tr>
<tr>
<td></td>
<td>C. What numbers did you place with the one whole? Why? What do you notice about the structure of fractions at the one whole mark?</td>
<td>C. Mathematics is critiqued and verified by people within particular cultures through justification or proof that is communicated to oneself and others.</td>
</tr>
<tr>
<td><strong>Grade 6</strong></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>A. How do the areas of resulting shapes compare to the area of the original shape?</td>
<td>A. Mathematics uses multiple strategies and multiple representations to make claims.</td>
</tr>
<tr>
<td></td>
<td>B. Can there be a formula for the area of a parallelogram that is only in terms of length of the sides (length times width)? Why?</td>
<td>B. Mathematics is a product of the exploration of structure and patterns.</td>
</tr>
<tr>
<td><strong>High School</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A. What do you notice about the graphs where $a &gt; 0$? Where $a &lt; 0$?</td>
<td>A. Mathematics is a product of the exploration of structure and patterns.</td>
</tr>
<tr>
<td></td>
<td>B. Is there similar behavior when graphing other types of parent functions?</td>
<td>B. Mathematics is critiqued and verified by people within particular cultures through justification and/or proof that is communicated to oneself and others.</td>
</tr>
<tr>
<td></td>
<td>C. How is a solution to a system of equations represented graphically? How can we find solutions algebraically?</td>
<td>C. Mathematics uses multiple strategies and multiple representations to make claims.</td>
</tr>
</tbody>
</table>

*Each ARC activity contains questions that provide opportunities to connect to the five-point view of NCTM.*
REFERENCES


